

Unified Framework of Mean-Field Formulations for Optimal Multi-period Mean-Variance Portfolio Selection*

Xiangyu Cui[†] Xun Li[‡] and Duan Li[§]

March 6, 2013

Abstract

The classical dynamic programming-based optimal stochastic control methods fail to cope with nonseparable dynamic optimization problems as the principle of optimality no longer applies in such situations. Among these notorious nonseparable problems, the dynamic mean-variance portfolio selection formulation had posted a great challenge to our research community until recently. A few solution methods, including the embedding scheme, have been developed in the last decade to solve the dynamic mean-variance portfolio selection formulation successfully. We propose in this paper a novel mean-field framework that offers a more efficient modeling tool and a more accurate solution scheme in tackling directly the issue of nonseparability and deriving the optimal policies analytically for the multi-period mean-variance-type portfolio selection problems.

KEY WORDS: Stochastic optimal control; mean-field formulation; multi-period portfolio selection; multi-period mean-variance formulation; intertemporal restrictions; risk control over bankruptcy.

1 Introduction

The mean-field type of optimal stochastic control models deals with problems in which both the system dynamics and objective functional could involve the states as well as the *expected values of the states*. The past few years have witnessed an increasing number of successful applications of the mean-field formulation, including mean-field type of stochastic control problems, in various fields of science, engineering, financial management, and economics. Although the research in this direction has been well developed for continuous-time control problems, it lacks progress in both theoretical investigation and applications in discrete-time problems. The current work in this paper aims to employ the mean-field formulation to cope with seemingly non-tractable nonseparability in discrete-time portfolio selection problems. In particular, we revisit three

*This work was partially supported by Research Grants Council of Hong Kong under grants 414207 and 520412, and by National Natural Science Foundation of China under grant 71201094.

[†]School of Statistics and Management, Shanghai University of Finance and Economics, Shanghai, China. E-mail: cui.xiangyu@mail.shufe.edu.cn.

[‡]Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong, China. E-mail: malixun@polyu.edu.hk.

[§]Corresponding author. Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Hong Kong, China. E-mail: dli@se.cuhk.edu.hk.

challenging, yet practically important, portfolio selection models over a finite-time investment horizon (see Li and Ng [16], Costa and Nabholz [10], Zhu et al. [28]), reformulate them as discrete-time linear-quadratic control problems of a mean-field type, and derive their optimal strategies with improved solution qualities.

Since Markowitz [18] published his seminal work on the mean-variance portfolio selection sixty years ago, the mean-risk portfolio selection framework has become one of the most significant ingredients in the modern financial theory. An important yet essential research theme under the mean-risk portfolio selection framework is to strike a balance between achieving a high mean of the investment return and minimizing the corresponding risk. If we adopt the variance of the terminal wealth as a risk measure for investment, we have the following mathematical formulations of the classical *static* mean-variance models,

$$\begin{aligned} (MV(\sigma)) \quad & \max \mathbb{E}(x_1), \\ & \text{s.t. } \text{Var}(x_1) \leq \sigma^2, \\ & x_1 = x_0 + u_0 \bullet S_0, \end{aligned}$$

and

$$\begin{aligned} (MV(\epsilon)) \quad & \min \text{Var}(x_1), \\ & \text{s.t. } \mathbb{E}(x_1) \geq \epsilon, \\ & x_1 = x_0 + u_0 \bullet S_0, \end{aligned}$$

which are equivalent to

$$\begin{aligned} (MV_s) \quad & \max \mathbb{E}(x_1) - \omega \text{Var}(x_1), \\ & \text{s.t. } x_1 = x_0 + u_0 \bullet S_0, \end{aligned}$$

where x_t is the wealth at time t , u_t is the portfolio strategy at time t , S_t is the random return at time t , $x_0 + u_0 \bullet S_0$ denotes the random terminal wealth x_1 from applying strategy u_0 in the market with initial wealth x_0 , and $\omega > 0$ denotes the trade-off between the two conflicting objectives of maximizing expected return and minimizing the risk. The optimal portfolio strategy and solution scheme of (MV_s) can be found in Merton [20] when shorting is allowed and in Markowitz [18] when shorting is prohibited.

However, the extension to a dynamic version of mean-variance portfolio selection was blocked for four decades until recently. Let us consider the following abstract form for the dynamic mean-variance portfolio selection problem,

$$\begin{aligned} (MV(\omega)) \quad & \max_u \mathbb{E}(x_T) - \omega \text{Var}(x_T), \\ & \text{s.t. } x_T = x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}, \end{aligned}$$

where $x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}$ denotes the random terminal wealth x_T from applying strategy $\{u_t\} \big|_{t=1}^{T-1}$ in the market with initial wealth x_0 . Due to the non-smoothing property of the variance term, i.e.,

$$\text{Var}(\text{Var}(\cdot | \mathcal{F}_i) | \mathcal{F}_j) \neq \text{Var}(\cdot | \mathcal{F}_j), \quad \forall i < j,$$

where \mathcal{F}_j is the information set available at time j and $\mathcal{F}_{j-1} \subset \mathcal{F}_j$, $(MV(\omega))$ is not a standard stochastic control problem whose objective functional involves the wealth state as well as a

nonlinear function of the expected wealth and, thus, does not satisfy the principle of optimality. Therefore, all the traditional dynamic programming-based optimal stochastic control solution methods no longer apply.

We now briefly summarize the main approaches in the current literature to overcome the difficulty resulted from the nonseparability. Adopting an embedding scheme, Li and Ng [16] and Zhou and Li [27] considered the following family of auxiliary problems, $\mathcal{A}(\omega, \lambda)$, parameterized in parameter λ ,

$$\begin{aligned} \mathcal{A}(\omega, \lambda) \quad & \min_u \mathbb{E}(\omega x_T^2 - \lambda x_T), \\ \text{s.t.} \quad & x_T = x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}. \end{aligned}$$

Note that problem $\mathcal{A}(\omega, \lambda)$ is a separable linear-quadratic stochastic control (LQSC) formulation and can be thus solved analytically. Li and Ng [16] and Zhou and Li [27] derived the optimal policy to the primal nonseparable problem $(MV(\omega))$ via identifying the optimal parameter λ and applying the optimal λ^* to $\mathcal{A}(\omega, \lambda)$. The embedding scheme has been also extended to multi-period mean-variance model with intertemporal restrictions (see Costa and Nabholz [10]), multi-period mean-variance model in a stochastic market whose evolution is governed by a Markovian chain (see Çelikyurt and Özekici [5]), a generalized mean-variance model with risk control over bankruptcy (see Zhu et al. [28]), and dynamic mean-variance asset-liability management (see Leippold et al. [15], Chiu and Li [9], Chen and Yang [8]).

By introducing an auxiliary variable d and an equality constraint $\mathbb{E}(x_T) = d$ for the expected terminal wealth, Li et al. [17] paved the road to study the following slightly modified, albeit equivalent, version of $(MV(\omega))$ (we omit the no-shorting constraint here and focus on the model itself),

$$\begin{aligned} (MV(d)) \quad & \min_u \text{Var}(x_T) = \mathbb{E}(x_T - d)^2, \\ \text{s.t.} \quad & \mathbb{E}(x_T) = d, \\ & x_T = x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}. \end{aligned}$$

Introducing a Lagrangian multiplier λ and applying Lagrangian relaxation to $(MV(d))$ gives rise to the following LQSC problem,

$$\begin{aligned} (L(\lambda)) \quad & \min \mathbb{E}(x_T - d)^2 - \lambda \mathbb{E}(x_T - d), \\ \text{s.t.} \quad & x_T = x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}. \end{aligned} \tag{1}$$

The optimal policy of $(MV(d))$ can be obtained by maximizing the dual function $L(\lambda)$ over all Lagrangian multiplier $\lambda \in \mathbb{R}$. In fact, the Lagrangian problem $(L(\lambda))$ can be further written as the following LQSC problem,

$$\begin{aligned} (MVH(m)) \quad & \min \mathbb{E}(x_T - m)^2, \\ \text{s.t.} \quad & x_T = x_0 + \{u_t \bullet S_t\} \big|_{t=0}^{T-1}, \end{aligned} \tag{2}$$

where $m = d + \lambda/2$. Problem $(MVH(m))$ is a special mean-variance hedging problem, in which an investor hedges the target m by his/her portfolio under a quadratic objective function. Problem $(MVH(m))$ has been well studied and can be solved by LQSC theory (see Li et al. [17]), martingale/convex duality theory (see Schweizer [23], Xia and Yan [25]) and sequential regression method (see Černý and Kellsen [6]).

In all the literature mentioned above, a static optimization procedure is always necessary to identify an optimal parameter in the parameterized auxiliary problem $\mathcal{A}(\omega, \lambda)$, $(L(\lambda))$ or $(MVH(m))$. Actually, based on the pure geometric structure of $(MV(\omega))$, Sun and Wang [24] proved that the optimal terminal wealth x_T^* takes the following form,

$$x_T^* = x_0 + \frac{1}{2\omega} \frac{1}{\mathbb{E}(1 - \{\varphi_t^* \bullet S_t\} \big|_{t=0}^{T-1})} \{\varphi_t^* \bullet S_t\} \big|_{t=0}^{T-1},$$

where φ^* is the policy of the following particular mean-variance hedging problem,

$$\begin{aligned} (MVH(1)) \quad & \min \quad \mathbb{E}(x_T - 1)^2, \\ \text{s.t.} \quad & x_T = \{\varphi_t \bullet S_t\} \big|_{t=0}^{T-1}. \end{aligned}$$

All the above approaches attempt to embed the “nontractable” nonseparable mean-variance portfolio selection problem into a family of tractable LQSC problems. Although these transformations seem necessary, one meaningful yet challenging question emerges naturally: Are we able to *directly* tackle the above nonseparable dynamic mean-variance problems (without introducing an auxiliary problem)?

The mean-variance problem is in fact a special case of the mean-field type problems where both the underlying dynamic system and the objective functional involve state processes as well as their expected values (hence the name mean-field). This critical feature differentiates the mean-variance problem from standard stochastic control problems. The theory of the *mean-field stochastic differential equation* can be traced back to Kac [14] who presented the McKean-Vlasov stochastic differential equation motivated by a stochastic toy model for the Vlasov kinetic equation of plasma. Since then, the research on related topics and their applications has become a notable and serious endeavor among researchers in applied probability and optimal stochastic controls, particularly in financial engineering. This new direction, however, requires new analytical tools and solution techniques. For instance, in a recent research on mean-field forward stochastic LQ optimal control problems, Yong [26] introduced a system of two Riccati equations to solve the problem. Representative works in mean-field include, but not limited to, McKean [19], Dawson [12], Chan [7], Buckdahn et al. [4], Borkar and Kumar [2], Crisan and Xiong [11], Andersson and Djehiche [1], Buckdahn et al. [3], Meyer-Brandis et al. [21], Nourian et al. [22] and Yong [26]. Despite active research efforts on mean-field in recent years, the topic of multi-period models in discrete-time remains a relatively unexplored subject where the mean-field modeling scheme has not yet been applied.

In this paper, we will develop a unified framework of mean-field formulations to investigate three multi-period mean-variance models in the literature: classical multi-period mean-variance model in Li and Ng [16], multi-period mean-variance model with intertemporal restrictions in Costa and Nabholz [10], and a generalized mean-variance model with risk control over bankruptcy in Zhu et al. [28]. We demonstrate that the mean-field approach represents a new promising way in dealing with nonseparable stochastic control problems related to the mean-variance formulations and even improves solution quality of some existing results in the literature.

2 Mean-Field Formulations for Multi-Period Mean-Variance Portfolio Selection

We consider in this paper a capital market consisting of one riskless asset and n risky assets within a time horizon T . Let s_t (> 1) be a given deterministic return of the riskless asset at period t and $\mathbf{e}_t = [e_t^1, \dots, e_t^n]'$ the vector of random returns of the n risky assets at period t . We assume that vectors \mathbf{e}_t , $t = 0, 1, \dots, T - 1$, are statistically independent and the only information known about the random return vector \mathbf{e}_t is its first two moments, its mean $\mathbb{E}(\mathbf{e}_t) = [\mathbb{E}(e_t^1), \mathbb{E}(e_t^2), \dots, \mathbb{E}(e_t^n)]'$ and its positive definite covariance

$$\text{Cov}(\mathbf{e}_t) = \mathbb{E}(\mathbf{e}_t \mathbf{e}_t') - \mathbb{E}(\mathbf{e}_t) \mathbb{E}(\mathbf{e}_t') = \begin{bmatrix} \sigma_{t,11} & \cdots & \sigma_{t,1n} \\ \vdots & \ddots & \vdots \\ \sigma_{t,n1} & \cdots & \sigma_{t,nn} \end{bmatrix} \succ 0.$$

From the above assumptions, we have

$$\begin{bmatrix} s_t^2 & s_t \mathbb{E}(\mathbf{e}_t') \\ s_t \mathbb{E}(\mathbf{e}_t) & \mathbb{E}(\mathbf{e}_t \mathbf{e}_t') \end{bmatrix} \succ 0.$$

We further define the excess return vector of risky assets \mathbf{P}_t as

$$\mathbf{P}_t = [P_t^1, P_t^2, \dots, P_t^n]' = [(e_t^1 - s_t), (e_t^2 - s_t), \dots, (e_t^n - s_t)]'.$$

The following is then true for $t = 0, 1, \dots, T - 1$:

$$\begin{bmatrix} s_t^2 & s_t \mathbb{E}(\mathbf{P}_t') \\ s_t \mathbb{E}(\mathbf{P}_t) & \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0}' \\ -\mathbf{1} & I \end{bmatrix} \begin{bmatrix} s_t^2 & s_t \mathbb{E}(\mathbf{e}_t') \\ s_t \mathbb{E}(\mathbf{e}_t) & \mathbb{E}(\mathbf{e}_t \mathbf{e}_t') \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0}' \\ -\mathbf{1} & I \end{bmatrix} \succ 0,$$

where $\mathbf{1}$ and $\mathbf{0}$ are the n -dimensional all-one and all-zero vectors, respectively, and I is the $n \times n$ identity matrix, which further implies

$$\begin{aligned} \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') &\succ 0, \quad \forall t = 0, 1, \dots, T - 1, \\ s_t^2(1 - B_t) &> 0, \quad \forall t = 0, 1, \dots, T - 1, \end{aligned}$$

where $B_t \triangleq \mathbb{E}(\mathbf{P}_t') \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)$.

An investor joins the market at the beginning of period 0 with an initial wealth x_0 . He/she allocates x_0 among the riskless asset and n risky assets at the beginning of period 0 and re-allocates his/her wealth at the beginning of each of the following $(T - 1)$ consecutive periods. Let x_t be the wealth of the investor at the beginning of period t , and u_t^i , $i = 1, 2, \dots, n$, be the amount invested in the i -th risky asset at period t . Then, $x_t - \sum_{i=1}^n u_t^i$ will be the amount invested in the riskless asset at period t . The information set at the beginning of period t is denoted as

$$\mathcal{F}_t = \sigma(\mathcal{F}_0 \vee \sigma(\mathbf{P}_0, \mathbf{P}_1, \dots, \mathbf{P}_{t-1})),$$

where \mathcal{F}_0 contains x_0 , s_t and the first and second moment information of \mathbf{P}_t , $t = 0, 1, \dots, T - 1$. We confine an admissible investment strategies to be \mathcal{F}_t -measurable Markov control, i.e., $\mathbf{u}_t \in \mathcal{F}_t$. Then, \mathbf{P}_t and \mathbf{u}_t are independent, $\{x_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(\mathcal{F}_0 \vee \sigma(x_t))$.

The conventional multi-period mean-variance model is to seek the best strategy, $\mathbf{u}_t^* = [(u_t^1)^*, (u_t^2)^*, \dots, (u_t^n)^*]'$, $t = 0, 1, \dots, T-1$, which is the optimizer of the following stochastic discrete-time optimal control problem,

$$(MMV) \quad \max \quad \mathbb{E}(x_T) - \omega_T \text{Var}(x_T), \quad (3)$$

$$\begin{aligned} \text{s.t.} \quad x_{t+1} &= \sum_{i=1}^n e_t^i u_t^i + \left(x_t - \sum_{i=1}^n u_t^i \right) s_t \\ &= s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \dots, T-1, \end{aligned} \quad (4)$$

where $\omega_T > 0$ is the trade-off parameter between the mean and the variance of the terminal wealth.

The multi-period mean-variance model with intertemporal restrictions is to find the optimal control of the following problem,

$$\begin{aligned} (MMV - IR) \quad \max \quad & \sum_{t \in I_\alpha} \alpha_t [\ell_t \mathbb{E}(x_t) - \rho_t \text{Var}(x_t)], \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \dots, T-1, \end{aligned}$$

where $I_\alpha = \{\tau_1, \dots, \tau_\alpha\}$ with $\tau_\alpha = T$ is the set of time instances on which the investor evaluates the performance of the portfolio, $\alpha_t \ell_t$ and $\alpha_t \rho_t > 0$ are the time- t weights of the mean and the variance in the objective functional. In particular, if we choose $I_\alpha = \{T\}$, $\alpha_T \ell_T = 1$ and $\alpha_T \rho_T = \omega_T > 0$, $(MMV - IR)$ reduces to the conventional multi-period mean-variance portfolio selection model (MMV) studied in Li and Ng [16]. If I_α contains time instances other than T , $(MMV - IR)$ is the multi-period portfolio selection problem with intertemporal restrictions considered in Costa and Nabholz [10]. Without loss of generality, we let I_α include all time instants from 0 to T , while setting some $\alpha_t = \ell_t = \rho_t = 0$ for these time instances which do not need to be evaluated.

The generalized mean-variance model for dynamic portfolio selection with risk control over bankruptcy is formulated as

$$\begin{aligned} (MMV - B) \quad \max \quad & \mathbb{E}(x_T) - \omega_T \text{Var}(x_T), \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \dots, T-1, \\ & P(x_t \leq b_t) \leq a_t, \quad t = 1, 2, \dots, T-1, \end{aligned}$$

where b_t is the disaster level and a_t is the acceptable maximum probability of bankruptcy set by the investor. By Tchebycheff inequality, problem $(MMV - B)$ can be transformed into the following (GMV) model (see Zhu et al. [28]),

$$\begin{aligned} (GMV) \quad \max \quad & \mathbb{E}(x_T) - \omega_T \text{Var}(x_T), \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \dots, T-1, \\ & \text{Var}(x_t) \leq a_t [\mathbb{E}(x_t) - b_t]^2, \quad t = 1, 2, \dots, T-1. \end{aligned}$$

To solve (GMV) , let us consider the Lagrangian maximization problem,

$$\begin{aligned} (L(\omega)) \quad \max \quad & \mathbb{E}(x_T) - \omega_T \text{Var}(x_T) - \sum_{t=1}^{T-1} \omega_t \left[\text{Var}(x_t) - a_t (\mathbb{E}(x_t) - b_t)^2 \right], \\ \text{s.t.} \quad & x_{t+1} = s_t x_t + \mathbf{P}_t' \mathbf{u}_t, \quad t = 0, 1, \dots, T-1, \end{aligned}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_{T-1})' \in \mathbb{R}_+^{T-1}$ is the vector of Lagrangian multipliers.

We are now building up the mean-field formulations for problems $(MMV - IR)$ and $(L(\omega))$, respectively. For $t = 0, 1, \dots, T-1$, the evolution of the expectation of the wealth dynamics specified in (4) can be presented as

$$\begin{cases} \mathbb{E}(x_{t+1}) = s_t \mathbb{E}(x_t) + \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t), \\ \mathbb{E}(x_0) = x_0, \end{cases} \quad (5)$$

due to the independence between \mathbf{P}_t and \mathbf{u}_t . Combining (4) and (5) yields the following for $t = 0, 1, \dots, T-1$,

$$\begin{cases} x_{t+1} - \mathbb{E}(x_{t+1}) = s_t(x_t - \mathbb{E}(x_t)) + \mathbf{P}'_t \mathbf{u}_t - \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t) \\ \quad = s_t(x_t - \mathbb{E}(x_t)) + \mathbf{P}'_t(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + (\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t)) \mathbb{E}(\mathbf{u}_t), \\ x_0 - \mathbb{E}(x_0) = 0. \end{cases} \quad (6)$$

What we are actually doing here is to enlarge the state space (x_t) into $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$ and the control space (\mathbf{u}_t) into $(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))$. Although control vector $\mathbb{E}(\mathbf{u}_t)$ and $\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)$ can be decided independently at time t , they should be chosen such that

$$\mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = \mathbf{0}, \quad t = 0, 1, \dots, T-1.$$

We also confine admissible investment strategies $(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))$ to be \mathcal{F}_t -measurable Markov control. Then, $\{(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))\}$ is again an adapted Markovian process and $\mathcal{F}_t = \sigma(\mathcal{F}_0 \vee \sigma(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)))$.

The problem $(MMV - IR)$ can be reformulated as a mean-field type of linear quadratic optimal stochastic control problem,

$$\begin{aligned} (MMV - MF) \quad & \max \sum_{t=1}^T \alpha_t \left\{ \ell_t \mathbb{E}(x_t) - \rho_t \mathbb{E}[(x_t - \mathbb{E}(x_t))^2] \right\}, \\ \text{s.t.} \quad & \mathbb{E}(x_t) \text{ satisfies dynamic equation (5),} \\ & x_{t+1} - \mathbb{E}(x_t) \text{ satisfies dynamic equation (6),} \\ & \mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = \mathbf{0}, \quad t = 0, 1, \dots, T-1. \end{aligned}$$

Similarly, problem $(L(\omega))$ can be reexpressed as

$$\begin{aligned} (L - MF(\omega)) \quad & \max \mathbb{E}(x_T) - \omega_T \mathbb{E}[(x_T - \mathbb{E}(x_T))^2] \\ & - \sum_{t=1}^{T-1} \omega_t \left\{ \mathbb{E}[(x_t - \mathbb{E}(x_t))^2] - a_t (\mathbb{E}(x_t) - b_t)^2 \right\}, \\ \text{s.t.} \quad & \mathbb{E}(x_t) \text{ satisfies dynamic equation (5),} \\ & x_{t+1} - \mathbb{E}(x_t) \text{ satisfies dynamic equation (6),} \\ & \mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = \mathbf{0}, \quad t = 0, 1, \dots, T-1. \end{aligned}$$

In the above two formulations of a mean-field type, the corresponding problems become separable linear quadratic optimal stochastic control problems in the expanded state space $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$ with the second control vector $\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)$ being constrained by a linear equation.

3 Optimal Policies for Multi-period Mean-Variance Portfolio Selection with and without Intertemporal Restrictions

Lemma 1 (Sherman-Morrison formula) Suppose that A is an invertible square matrix and μ and ν are two given vectors. If

$$1 + \nu' A^{-1} \mu \neq 0,$$

then the following holds,

$$(A + \mu \nu')^{-1} = A^{-1} - \frac{A^{-1} \mu \nu' A^{-1}}{1 + \nu' A^{-1} \mu}.$$

Lemma 2 Let $B_t = \mathbb{E}(\mathbf{P}_t' \mathbf{P}_t) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)$. Then

$$[\mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t')]^{-1} \mathbb{E}(\mathbf{P}_t) = \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t}.$$

Proof. Applying Sherman-Morrison formula gives rise to the following,

$$\begin{aligned} & [\mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t')]^{-1} \mathbb{E}(\mathbf{P}_t) \\ &= \left[\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') + \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t') \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t')}{1 - \mathbb{E}(\mathbf{P}_t') \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)} \right] \mathbb{E}(\mathbf{P}_t) \\ &= \left[\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') + \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t') \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t')}{1 - B_t} \right] \mathbb{E}(\mathbf{P}_t) \\ &= \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t}. \end{aligned}$$

□

Consider the following separable multi-period control problem,

$$\begin{aligned} \max \quad & \mathbb{E} \left[\sum_{t=0}^{T-1} h_t(x_t, v_t) + h_T(x_T) \right], \\ \text{s.t.} \quad & x_{t+1} = f(x_t, v_t), \quad t = 0, 1, \dots, T-1, \end{aligned}$$

where x_t denotes the state, v_t denotes the control, $f(x_t, v_t)$ represents the dynamics of the state and $h_t(x_t, v_t)$ is concave in v_t . Based on the principle of optimality in dynamic programming, the optimal control at time t is derived from the following recursion of dynamic programming,

$$v_t^* = \operatorname{argmax}_{v_t} \{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) | \mathcal{F}_t] + h_t(x_t, v_t) \},$$

where \mathcal{F}_t is the information set at time t , (v_0, v_1, \dots, v_t) is the control sequence before time $t+1$ and

$$J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) = \max_{v_{t+1}, \dots, v_{T-1}} \mathbb{E} \left[\sum_{j=t+1}^{T-1} h_j(x_j, v_j) + h_T(x_T) \middle| \mathcal{F}_{t+1} \right]$$

is the benefit-to-go function at time $t+1$.

Lemma 3 Assume that

$$\mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) | \mathcal{F}_t] = G_t^1(x_t; v_0, v_1, \dots, v_t) + G_t^2(x_t; v_0, v_1, \dots, v_t),$$

where $\mathbb{E}[G_t^2(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] = 0$ holds for any admissible (v_0, v_1, \dots, v_t) . Then

$$\begin{aligned} v_t^* &= \operatorname{argmax}_{v_t} \left\{ G_t^1(x_t; v_0, v_1, \dots, v_t) + h_t(x_t, v_t) \right\}, \\ J_0(x_0) &= \max_{v_0, \dots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\}, \\ &\quad t = 0, 1, \dots, T-1, \end{aligned}$$

i.e., $G_t^1(x_t; v_0, v_1, \dots, v_t^*) + h_t(x_t, v_t^*)$ can be regarded as the benefit-to-go function at time t .

Proof. Based on the principle of optimality of dynamic programming, the optimal control sequence on or before time $t+1$ is determined by

$$(v_0^*, v_1^*, \dots, v_t^*) = \operatorname{argmax}_{v_0, \dots, v_t} \left\{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\}.$$

Thus, we have

$$\begin{aligned} &(v_0^*, v_1^*, \dots, v_t^*) \\ &= \operatorname{argmax}_{v_0, \dots, v_t} \left\{ \mathbb{E}[\mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) | \mathcal{F}_t] | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\} \\ &= \operatorname{argmax}_{v_0, \dots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) + G_t^2(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\} \\ &= \operatorname{argmax}_{v_0, \dots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\} \\ &= \operatorname{argmax}_{v_0, \dots, v_t} \left\{ \mathbb{E}[\dots \mathbb{E}[\mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) + h_t(x_t, v_t) | \mathcal{F}_{t-1}] + h_{t-1}(x_{t-1}, v_{t-1}) | \mathcal{F}_{t-2}] \dots | \mathcal{F}_0] \right. \\ &\quad \left. + h_0(x_0, v_0) \right\}, \end{aligned}$$

which implies

$$v_t^* = \operatorname{argmax}_{v_t} \left\{ G_t^1(x_t; v_0, v_1, \dots, v_t) + h_t(x_t, v_t) \right\}.$$

Since $\mathbb{E}[G_t^2(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] = 0$ holds for any admissible (v_0, v_1, \dots, v_t) , we have

$$\begin{aligned} J_0(x_0) &= \max_{v_0, \dots, v_t} \left\{ \mathbb{E}[J_{t+1}(x_{t+1}; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\} \\ &= \max_{v_0, \dots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j, v_j) | \mathcal{F}_0] \right\}. \end{aligned}$$

□

Remark 1 Please note that if $h_t(x_t, v_t) = h_t(x_t)$, i.e., h_t is independent of control v_t , the conclusion of Lemma 3 can be expressed as follows,

$$v_t^* = \operatorname{argmax}_{v_t} G_t^1(x_t; v_0, v_1, \dots, v_t),$$

$$J_0(x_0) = \max_{v_0, \dots, v_t} \left\{ \mathbb{E}[G_t^1(x_t; v_0, v_1, \dots, v_t) | \mathcal{F}_0] + \sum_{j=0}^t \mathbb{E}[h_j(x_j) | \mathcal{F}_0] \right\},$$

$$t = 0, 1, \dots, T-1,$$

i.e., $G_t^1(x_t; v_0, v_1, \dots, v_t^*) + h_t(x_t)$ can be regarded as the benefit-to-go function at time t .

In this section, we reconsider the classical multi-period mean-variance model in Li and Ng [16] and the multi-period mean-variance model with intertemporal restrictions, $(MMV - MF)$, in Costa and Nabholz [10] under a mean-field formulation. Before presenting our main proposition, we define the following backwards recursions for p_t and q_t ,

$$\begin{cases} p_t = \alpha_t \rho_t + s_t^2(1 - B_t)p_{t+1}, & \begin{cases} q_t = \alpha_t \ell_t + s_t q_{t+1}, \\ q_T = \alpha_T \ell_T, \end{cases} \\ p_T = \alpha_T \rho_T, \end{cases}$$

for $t = T-1, T-2, \dots, 1$. We also set $\prod_{\emptyset}(\cdot) = 1$ and $\sum_{\emptyset}(\cdot) = 0$ for the convenience.

Proposition 1 The optimal strategy of problem $(MMV - MF)$ is given by

$$\mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*) = -s_t(x_t - \mathbb{E}(x_t))\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t')\mathbb{E}(\mathbf{P}_t), \quad (7)$$

$$\mathbb{E}(\mathbf{u}_t^*) = \frac{q_{t+1}}{2p_{t+1}} \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t')\mathbb{E}(\mathbf{P}_t)}{1 - B_t}, \quad (8)$$

for $t = 0, 1, \dots, T-1$, where the optimal expected wealth level is

$$\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} s_k + \sum_{j=0}^{t-1} \frac{q_{j+1}}{2p_{j+1}} \cdot \frac{B_j}{1 - B_j} \cdot \prod_{\ell=j+1}^{t-1} s_\ell.$$

Proof. We first prove that, for information set $\mathcal{F}_t = \sigma(\mathcal{F}_0 \vee \sigma(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)))$, we have the following expression,

$$J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) = -p_t(x_t - \mathbb{E}(x_t))^2 + q_t \mathbb{E}(x_t) + \sum_{j=t}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j, \quad (9)$$

as the benefit-to-go function at time t .

When $t = T$, expression (9) is obvious. Assume that we have expression (9) as the benefit-to-go function at time $t+1$. We prove that expression (9) still holds for the benefit-to-go function at time t . For given information set \mathcal{F}_t , i.e., $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$, the recursive equation reads as

$$\begin{aligned} & J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) \\ &= -\alpha_t \rho_t (x_t - \mathbb{E}(x_t))^2 + \alpha_t \ell_t \mathbb{E}(x_t) + \max_{(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))} \mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t]. \end{aligned}$$

Based on dynamics (5) and (6), we deduce

$$\begin{aligned}
& \mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t] \\
&= \mathbb{E}[-p_{t+1}(x_{t+1} - \mathbb{E}(x_{t+1}))^2 + q_{t+1}\mathbb{E}(x_{t+1}) | \mathcal{F}_t] + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j \\
&= -p_{t+1}\mathbb{E}\left[s_t^2(x_t - \mathbb{E}(x_t))^2 + \left(\mathbf{P}'_t(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right)^2 + \left(\mathbb{E}(\mathbf{u}'_t)(\mathbf{P}_t - \mathbb{E}(\mathbf{P}_t))\right)^2\right. \\
&\quad \left.+ 2s_t(x_t - \mathbb{E}(x_t))\mathbf{P}'_t(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + 2s_t(x_t - \mathbb{E}(x_t))(\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t)\right. \\
&\quad \left.+ 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'\mathbf{P}_t(\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \Big| \mathcal{F}_t\right] + q_{t+1}[s_t\mathbb{E}(x_t) + \mathbb{E}(\mathbf{P}'_t)\mathbb{E}(\mathbf{u}_t)] + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j.
\end{aligned}$$

Since both $\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)$ and $\mathbb{E}(\mathbf{u}_t)$ are \mathcal{F}_t -measurable and \mathbf{P}_t is independent to \mathcal{F}_t , we have

$$\begin{aligned}
& \mathbb{E}\left[\left(\mathbf{P}'_t(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right)^2 \Big| \mathcal{F}_t\right] = (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)), \\
& \mathbb{E}\left[\left(\mathbb{E}(\mathbf{u}'_t)(\mathbf{P}_t - \mathbb{E}(\mathbf{P}_t))\right)^2 \Big| \mathcal{F}_t\right] = \mathbb{E}(\mathbf{u}'_t)(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t), \\
& \mathbb{E}\left[2s_t(x_t - \mathbb{E}(x_t))\mathbf{P}'_t(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\mathbb{E}(\mathbf{u}_t) \Big| \mathcal{F}_t\right] = 2s_t(x_t - \mathbb{E}(x_t))\mathbb{E}(\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)), \\
& \mathbb{E}\left[2s_t(x_t - \mathbb{E}(x_t))(\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \Big| \mathcal{F}_t\right] = 0, \\
& \mathbb{E}\left[2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'\mathbf{P}_t(\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \Big| \mathcal{F}_t\right] = 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t),
\end{aligned}$$

which further implies,

$$\begin{aligned}
& \mathbb{E}[J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t] \\
&= -p_{t+1}\left[s_t^2(x_t - \mathbb{E}(x_t))^2 + (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right. \\
&\quad \left.+ 2s_t(x_t - \mathbb{E}(x_t))\mathbb{E}(\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right] - p_{t+1}\mathbb{E}(\mathbf{u}'_t)(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \\
&\quad + q_{t+1}\mathbb{E}(\mathbf{P}'_t)\mathbb{E}(\mathbf{u}_t) + s_t q_{t+1}\mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j \\
&\quad + 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \\
&= G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)),
\end{aligned}$$

where

$$\begin{aligned}
& G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \\
&= -p_{t+1}\left[s_t^2(x_t - \mathbb{E}(x_t))^2 + (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right. \\
&\quad \left.+ 2s_t(x_t - \mathbb{E}(x_t))\mathbb{E}(\mathbf{P}'_t)(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))\right] - p_{t+1}\mathbb{E}(\mathbf{u}'_t)(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t) \\
&\quad + q_{t+1}\mathbb{E}(\mathbf{P}'_t)\mathbb{E}(\mathbf{u}_t) + s_t q_{t+1}\mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j \\
& G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))'(\mathbb{E}(\mathbf{P}_t\mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}'_t))\mathbb{E}(\mathbf{u}_t).
\end{aligned}$$

Note that any admissible $(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))$ satisfies $\mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = 0$, which implies

$$\begin{aligned} & \mathbb{E}[G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) | \mathcal{F}_0] \\ &= 2\mathbb{E}[(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' (\mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t')) \mathbb{E}(\mathbf{u}_t) | \mathcal{F}_0] = 0. \end{aligned}$$

Using Lemma 3 and Remark 1, we get

$$(\mathbb{E}(\mathbf{u}_t^*), \mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*)) = \operatorname{argmax} G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)).$$

By means of Lemma 2, we deduce

$$\begin{aligned} & G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \\ &= -p_{t+1} \left\{ s_t^2 (1 - B_t) (x_t - \mathbb{E}(x_t))^2 + \left[(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \right]' \right. \\ & \quad \cdot \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') \left[(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \right] \left. \right\} \\ & \quad - p_{t+1} \left[\mathbb{E}(\mathbf{u}_t) - \frac{q_{t+1}}{2p_{t+1}} \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t} \right]' (\mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t')) \\ & \quad \cdot \left[\mathbb{E}(\mathbf{u}_t) - \frac{q_{t+1}}{2p_{t+1}} \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t} \right] + \frac{q_{t+1}^2}{4p_{t+1}} B_t + s_t q_{t+1} \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*) &= -s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t), \\ \mathbb{E}(\mathbf{u}_t^*) &= \frac{q_{t+1}}{2p_{t+1}} \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t}, \end{aligned}$$

where the linear constraint $\mathbb{E}(\mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*)) = \mathbf{0}$ automatically holds. Therefore, based on Remark 1, we have

$$\begin{aligned} & G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t^*), \mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*)) - \alpha_t \rho_t (x_t - \mathbb{E}(x_t))^2 + \alpha_t \ell_t \mathbb{E}(x_t) \\ &= -p_t (x_t - \mathbb{E}(x_t))^2 + q_t \mathbb{E}(x_t) + \sum_{j=t}^{T-1} \frac{q_{j+1}^2}{4p_{j+1}} B_j \end{aligned}$$

as the benefit-to-go function at time t .

Substituting the optimal expected portfolio strategy (8) into dynamics (5), we further deduce the following recursive relationship of the optimal expected wealth level,

$$\mathbb{E}(x_{t+1}) = s_t \mathbb{E}(x_t) + \frac{q_{t+1}}{2p_{t+1}} \cdot \frac{B_t}{1 - B_t},$$

which implies

$$\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} s_k + \sum_{j=0}^{t-1} \frac{q_{j+1}}{2p_{j+1}} \cdot \frac{B_j}{1 - B_j} \cdot \prod_{\ell=j+1}^{t-1} s_\ell.$$

□

The optimal strategy obtained in Proposition 1 covers the exiting results in the literature as its special cases.

Case 1: Let $I_\alpha = \{T\}$, $\alpha_T \ell_T = 1$, $\alpha_T \rho_T = \omega_T > 0$. Then, we have

$$p_t = \omega_T \prod_{j=t}^{T-1} s_j^2 (1 - B_j), \quad q_t = \prod_{j=t}^{T-1} s_j,$$

which further implies

$$\begin{aligned} \mathbb{E}(x_t) &= \prod_{k=0}^{t-1} s_k x_0 + \frac{1}{2\omega_T} \prod_{k=t}^{T-1} s_k^{-1} \sum_{j=0}^{t-1} B_j \prod_{\ell=j}^{T-1} (1 - B_\ell)^{-1} \\ &= \prod_{k=0}^{t-1} s_k x_0 + \frac{1}{2\omega_T} \prod_{k=t}^{T-1} s_k^{-1} \frac{1 - \prod_{k=0}^{t-1} (1 - B_k)}{\prod_{k=0}^{T-1} (1 - B_k)}, \\ \mathbb{E}(x_T) &= \prod_{k=0}^{T-1} s_k x_0 + \frac{1}{2\omega_T} \cdot \frac{1 - \prod_{k=0}^{T-1} (1 - B_k)}{\prod_{k=0}^{T-1} (1 - B_k)}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbf{u}_t^* &= -s_t \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') x_t + s_t \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \mathbb{E}(x_t) + \mathbb{E}(u_t^*) \\ &= -s_t \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') x_t + \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \left[x_0 \prod_{k=0}^{T-1} s_k + \frac{1}{2\omega_T \prod_{k=0}^{T-1} (1 - B_k)} \right] \prod_{k=t+1}^{T-1} s_k^{-1}, \end{aligned} \quad (10)$$

which is the optimal portfolio strategy obtained in Li and Ng [16].

Substituting (8) and (10) to dynamics (6) yields

$$\mathbb{E}(x_{t+1} - \mathbb{E}(x_{t+1}))^2 = s_t^2 (1 - B_t) \mathbb{E}(x_t - \mathbb{E}(x_t))^2 + \frac{1}{\prod_{k=t+1}^{T-1} s_k^2 (1 - B_k)} \cdot \frac{B_t}{4\omega_T^2 \prod_{k=t}^{T-1} (1 - B_j)},$$

which further implies

$$\begin{aligned} \mathbb{E}(x_T - \mathbb{E}(x_T))^2 &= \sum_{j=0}^{T-1} \prod_{k=j+1}^{T-1} s_k^2 (1 - B_k) \frac{1}{\prod_{k=j+1}^{T-1} s_k^2 (1 - B_k)} \cdot \frac{B_j}{4\omega_T^2 \prod_{k=j}^{T-1} (1 - B_j)} \\ &= \frac{1}{4\omega_T^2} \sum_{j=0}^{T-1} B_j \prod_{k=j}^{T-1} (1 - B_k)^{-1} \\ &= \frac{1 - \prod_{k=0}^{T-1} (1 - B_k)}{4\omega_T^2 \prod_{k=0}^{T-1} (1 - B_k)}. \end{aligned}$$

Thus, the efficient frontier is given by

$$\text{Var}(x_T) = \mathbb{E}(x_T - \mathbb{E}(x_T))^2 = \frac{\prod_{k=0}^{T-1} (1 - B_k)}{1 - \prod_{k=0}^{T-1} (1 - B_k)} \left(\mathbb{E}(x_T) - x_0 \prod_{k=0}^{T-1} s_k \right)^2 \quad \text{for } \mathbb{E}(x_T) \geq x_0 \prod_{k=0}^{T-1} s_k,$$

which is the same as the efficient frontier established in Li and Ng [16].

Case 2: Let $I_\alpha = \{\tau_1, \dots, \tau_\alpha\}$ with $\tau_\alpha = T$. Then we have the optimal portfolio strategy as follows,

$$\mathbf{u}_t^* = -s_t x_t \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) + s_t \mathbb{E}(x_t) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) + \frac{q_{t+1}}{2p_{t+1}} \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{1 - B_t}, \quad (11)$$

where

$$\begin{cases} p_t = \alpha_t \rho_t + s_t^2 (1 - B_t) p_{t+1}, & t = \tau_i, \\ p_t = s_t^2 (1 - B_t) p_{t+1}, & \tau_{i-1} < t < \tau_i, \\ p_T = \alpha_T \rho_T, \end{cases} \quad \begin{cases} q_t = \alpha_t \ell_t + s_t q_{t+1}, & t = \tau_i, \\ q_t = s_t q_{t+1}, & \tau_{i-1} < t < \tau_i, \\ q_T = \alpha_T \ell_T, \end{cases}$$

and

$$\mathbb{E}(x_{t+1}) = s_t \mathbb{E}(x_t) + \frac{q_{t+1}}{2p_{t+1}} \frac{B_t}{1 - B_t},$$

which is the same as the result developed in Costa and Nabholz [10]. Note that Costa and Nabholz originally studied a market consisting of all risky assets in their investigation. When we introduce a riskless asset into the market, parameters of \mathcal{G}_i , \mathcal{S}_i , \mathcal{A}_i and \mathcal{D}_i defined in (22), (23), (28) and (29), respectively, in Costa and Nabholz [10] have been modified to

$$\mathcal{G}_i = -2p_{\tau_i}, \quad \mathcal{S}_i = -q_{\tau_i}, \quad \mathcal{A}_i = \prod_{k=\tau_i}^{\tau_{i+1}-1} s_k, \quad \mathcal{D}_i = \frac{1 - \prod_{k=\tau_i}^{\tau_{i+1}-1} (1 - B_k)}{\prod_{k=\tau_i}^{\tau_{i+1}-1} (1 - B_k)} \cdot \frac{q_{\tau_{i+1}}}{2p_{\tau_{i+1}}}.$$

4 Generalized Mean-Variance Strategy with Risk Control Over Bankruptcy

In this section, we reconsider the generalized mean-variance model with risk control over bankruptcy in Zhu et al. [28] under the mean-field framework, i.e., we consider problem $(L - MF(\omega))$ first. For $t = T - 1, T - 2, \dots, 1$, we define \bar{p}_t , η_t and ξ_t as follows,

$$\begin{cases} \bar{p}_t = \omega_t + s_t^2 (1 - B_t) \bar{p}_{t+1}, \\ \bar{p}_T = \omega_T, \end{cases} \quad \begin{cases} \eta_t = \omega_t a_t + s_t^2 \zeta_{t+1} \eta_{t+1}, \\ \eta_T = 0, \end{cases} \quad \begin{cases} \xi_t = -\omega_t a_t b_t + s_t \zeta_{t+1} \xi_{t+1}, \\ \xi_T = \frac{1}{2}, \end{cases}$$

where Lagrangian multiplier $\omega_t \geq 0$ and

$$\zeta_{t+1} = \frac{\bar{p}_{t+1}(1 - B_t) + 2\eta_{t+1}B_t}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1}B_t} \geq 0,$$

due to $1 > B_t > 0$. Then, it is obvious that $\bar{p}_t > 0$ and $\eta_t \geq 0$.

Lemma 4 Suppose that $\bar{p}_{t+1} > 0$ and $\eta_{t+1} \geq 0$ hold. Then

$$[\bar{p}_{t+1} \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - (\bar{p}_{t+1} + \eta_{t+1}) \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t')]^{-1} \mathbb{E}(\mathbf{P}_t) = \frac{\mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1}B_t}.$$

Proof. Applying Sherman-Morrison formula (Lemma 1) yields

$$\begin{aligned}
& [\bar{p}_{t+1}\mathbb{E}(\mathbf{P}_t\mathbf{P}_t') - (\bar{p}_{t+1} + \eta_{t+1})\mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}_t')]^{-1}\mathbb{E}(\mathbf{P}_t) \\
&= \left[\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t') + \frac{\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')(\bar{p}_{t+1} + \eta_{t+1})\mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}_t')\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')}{1 - \bar{p}_{t+1}^{-1}(\bar{p}_{t+1} + \eta_{t+1})\mathbb{E}(\mathbf{P}_t')\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')\mathbb{E}(\mathbf{P}_t)} \right] \mathbb{E}(\mathbf{P}_t) \\
&= \left[\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t') + \frac{\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')(\bar{p}_{t+1} + \eta_{t+1})\mathbb{E}(\mathbf{P}_t)\mathbb{E}(\mathbf{P}_t')\bar{p}_{t+1}^{-1}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')}{1 - \bar{p}_{t+1}^{-1}(\bar{p}_{t+1} + \eta_{t+1})B_t} \right] \mathbb{E}(\mathbf{P}_t) \\
&= \frac{\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')\mathbb{E}(\mathbf{P}_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1}B_t}.
\end{aligned}$$

□

Proposition 2 *The optimal strategy of problem $(L - MF(\omega))$ is given by*

$$\mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*) = -s_t(x_t - \mathbb{E}(x_t))\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')\mathbb{E}(\mathbf{P}_t), \quad (12)$$

$$\mathbb{E}(\mathbf{u}_t^*) = \frac{\xi_{t+1} + \eta_{t+1}s_t\mathbb{E}(x_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1}B_t}\mathbb{E}^{-1}(\mathbf{P}_t\mathbf{P}_t')\mathbb{E}(\mathbf{P}_t), \quad (13)$$

where the optimal expected wealth level $\mathbb{E}(x_t)$ evolves according to

$$\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} \zeta_{k+1}s_k + \sum_{j=0}^{t-1} \frac{\xi_{j+1}B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1}B_j} \cdot \prod_{\ell=j+1}^{t-1} \zeta_{\ell+1}s_\ell. \quad (14)$$

Moreover, the optimal objective function of $(L - MF(\omega))$ is

$$H(\omega) = \eta_1\zeta_1s_0^2x_0^2 + 2\xi_1\zeta_1s_0x_0 + \sum_{j=0}^{T-1} \left[\frac{\xi_{j+1}^2B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1}B_j} + \omega_ja_jb_j^2 \right], \quad (15)$$

with $\omega_0 = a_0 = b_0 = 0$.

Proof. We first prove that for information set $\mathcal{F}_t = \sigma(\mathcal{F}_0 \vee \sigma(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)))$, we have the following expression,

$$\begin{aligned}
& J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) \\
&= -\bar{p}_t(x_t - \mathbb{E}(x_t))^2 + \eta_t(\mathbb{E}(x_t))^2 + 2\xi_t\mathbb{E}(x_t) + \sum_{j=t}^{T-1} \left[\frac{\xi_{j+1}^2B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1}B_j} + \omega_ja_jb_j^2 \right], \quad (16)
\end{aligned}$$

as the benefit-to-go function at time t .

When $t = T$, expression (16) is obvious. Assume that expression (16) holds at time $t + 1$ as the benefit-to-go function. We show that expression (16) still holds for the benefit-to-go function at time t . For given information set \mathcal{F}_t , i.e., $(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t))$, applying the recursive equation yields

$$\begin{aligned}
J_t(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t)) &= -\omega_t(x_t - \mathbb{E}(x_t))^2 + \omega_ta_t(\mathbb{E}(x_t))^2 - 2\omega_ta_tb_t\mathbb{E}(x_t) + \omega_ta_tb_t^2 \\
&\quad + \max_{(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))} \mathbb{E} [J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t].
\end{aligned}$$

Based on the dynamics in (5) and (6), we have

$$\begin{aligned}
& \mathbb{E} [J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t] \\
&= \mathbb{E} \left[-\bar{p}_{t+1} (x_{t+1} - \mathbb{E}(x_{t+1}))^2 + \eta_{t+1} (\mathbb{E}(x_{t+1}))^2 + 2\xi_{t+1} \mathbb{E}(x_{t+1}) | \mathcal{F}_t \right] \\
& \quad + \sum_{j=t+1}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1-B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right] \\
&= -\bar{p}_{t+1} \mathbb{E} \left[s_t^2 (x_t - \mathbb{E}(x_t))^2 + \left(\mathbf{P}'_t (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \right)^2 + \left(\mathbb{E}(\mathbf{u}'_t) (\mathbf{P}_t - \mathbb{E}(\mathbf{P}_t)) \right)^2 \right. \\
& \quad + 2s_t (x_t - \mathbb{E}(x_t)) \mathbf{P}'_t (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + 2s_t (x_t - \mathbb{E}(x_t)) (\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t)) \mathbb{E}(\mathbf{u}_t) \\
& \quad \left. + 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' \mathbf{P}_t (\mathbf{P}'_t - \mathbb{E}(\mathbf{P}'_t)) \mathbb{E}(\mathbf{u}_t) | \mathcal{F}_t \right] + \eta_{t+1} \left[s_t \mathbb{E}(x_t) + \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t) \right]^2 \\
& \quad + 2\xi_{t+1} \left[s_t \mathbb{E}(x_t) + \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t) \right] + \sum_{j=t+1}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1-B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right].
\end{aligned}$$

Similar to the proof of Proposition 1, we have

$$\begin{aligned}
& \mathbb{E} [J_{t+1}(\mathbb{E}(x_{t+1}), x_{t+1} - \mathbb{E}(x_{t+1})) | \mathcal{F}_t] \\
&= -\bar{p}_{t+1} \left[s_t^2 (x_t - \mathbb{E}(x_t))^2 + (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' \mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \right. \\
& \quad \left. + 2s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}(\mathbf{P}'_t) (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \right] - \mathbb{E}(\mathbf{u}'_t) \left[\bar{p}_{t+1} \mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) - (\bar{p}_{t+1} + \eta_{t+1}) \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}'_t) \right] \mathbb{E}(\mathbf{u}_t) \\
& \quad + (2\xi_{t+1} + 2\eta_{t+1} s_t \mathbb{E}(x_t)) \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t) + \eta_{t+1} s_t^2 (\mathbb{E}(x_t))^2 + 2\xi_{t+1} s_t \mathbb{E}(x_t) \\
& \quad + \sum_{j=t+1}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1-B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right] + 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' (\mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}'_t)) \mathbb{E}(\mathbf{u}_t) \\
&= G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)),
\end{aligned}$$

where

$$\begin{aligned}
& G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \\
&= -\bar{p}_{t+1} \left[s_t^2 (x_t - \mathbb{E}(x_t))^2 + (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' \mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \right. \\
& \quad \left. + 2s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}(\mathbf{P}'_t) (\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \right] - \mathbb{E}(\mathbf{u}'_t) \left[\bar{p}_{t+1} \mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) - (\bar{p}_{t+1} + \eta_{t+1}) \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}'_t) \right] \mathbb{E}(\mathbf{u}_t) \\
& \quad + (2\xi_{t+1} + 2\eta_{t+1} s_t \mathbb{E}(x_t)) \mathbb{E}(\mathbf{P}'_t) \mathbb{E}(\mathbf{u}_t) + \eta_{t+1} s_t^2 (\mathbb{E}(x_t))^2 + 2\xi_{t+1} s_t \mathbb{E}(x_t) \\
& \quad + \sum_{j=t+1}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1-B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right], \\
& G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = 2(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))' (\mathbb{E}(\mathbf{P}_t \mathbf{P}'_t) - \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}'_t)) \mathbb{E}(\mathbf{u}_t).
\end{aligned}$$

Note that any admissible $(\mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t))$ satisfies $\mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = 0$, which implies

$$\mathbb{E} [G_t^2(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) | \mathcal{F}_0] = 0.$$

Using Lemma 3 and corresponding to Remark 1, we get

$$(\mathbb{E}(\mathbf{u}_t^*), \mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*)) = \operatorname{argmax} G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)).$$

By means of Lemma 2, we deduce

$$\begin{aligned}
& G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t), \mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) \\
&= -\bar{p}_{t+1} \mathbb{E} \left\{ s_t^2 (1 - B_t) (x_t - \mathbb{E}(x_t))^2 + \left[(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \right]' \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') \right. \\
&\quad \cdot \left. \left[(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) + s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t') \right] \right\} \\
&\quad - \left[\mathbb{E}(\mathbf{u}_t) - \frac{\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t} \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) \right]' \left[\bar{p}_{t+1} \mathbb{E}(\mathbf{P}_t \mathbf{P}_t') - (\bar{p}_{t+1} + \eta_{t+1}) \mathbb{E}(\mathbf{P}_t) \mathbb{E}(\mathbf{P}_t') \right] \\
&\quad \cdot \left[\mathbb{E}(\mathbf{u}_t) - \frac{\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t} \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) \right] + \frac{(\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t))^2}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t} B_t \\
&\quad + \eta_{t+1} s_t^2 (\mathbb{E}(x_t))^2 + 2\xi_{t+1} s_t \mathbb{E}(x_t) + \sum_{j=t+1}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*) &= -s_t (x_t - \mathbb{E}(x_t)) \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t), \\
\mathbb{E}(\mathbf{u}_t^*) &= \frac{\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t)}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t} \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t),
\end{aligned}$$

which satisfies the linear constraint $\mathbb{E}(\mathbf{u}_t - \mathbb{E}(\mathbf{u}_t)) = \mathbf{0}$.

Based on Remark 1, we can find

$$\begin{aligned}
& G_t^1(\mathbb{E}(x_t), x_t - \mathbb{E}(x_t); \mathbb{E}(\mathbf{u}_t^*), \mathbf{u}_t^* - \mathbb{E}(\mathbf{u}_t^*)) \\
&\quad - \omega_t (x_t - \mathbb{E}(x_t))^2 + \omega_t a_t (\mathbb{E}(x_t))^2 - 2\omega_t a_t b_t \mathbb{E}(x_t) + \omega_t a_t b_t^2 \\
&= -\bar{p}_t (x_t - \mathbb{E}(x_t))^2 + \eta_t (\mathbb{E}(x_t))^2 + 2\xi_t \mathbb{E}(x_t) + \sum_{j=t}^{T-1} \left[\frac{\xi_{j+1}^2 B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1} B_j} + \omega_j a_j b_j^2 \right]
\end{aligned}$$

as the benefit-to-go function at time t .

Substituting the optimal expected portfolio strategy (13) into dynamics (5) gives rise to

$$\mathbb{E}(x_{t+1}) = \zeta_{t+1} s_t \mathbb{E}(x_t) + \frac{\xi_{t+1} B_t}{\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t},$$

which implies

$$\mathbb{E}(x_t) = x_0 \prod_{k=0}^{t-1} \zeta_{k+1} s_k + \sum_{j=0}^{t-1} \frac{\xi_{j+1} B_j}{\bar{p}_{j+1}(1 - B_j) + \eta_{j+1} B_j} \cdot \prod_{\ell=j+1}^{t-1} \zeta_{\ell+1} s_\ell.$$

Noting that $\omega_0 = a_0 = b_0 = 0$, then the expression of the optimal objective function of $(L - MF(\omega))$ is obvious. \square

Substituting (12) and (13) to dynamics (6) yields

$$\mathbb{E}(x_{t+1} - \mathbb{E}(x_{t+1}))^2 = s_t^2 (1 - B_t) \mathbb{E}(x_t - \mathbb{E}(x_t))^2 + \frac{(\xi_{t+1} + \eta_{t+1} s_t \mathbb{E}(x_t))^2}{(\bar{p}_{t+1}(1 - B_t) + \eta_{t+1} B_t)^2} (B_t - B_t^2),$$

which leads to the following expression of the variance of the optimal wealth level,

$$\text{Var}(x_t) = \mathbb{E}(x_t - \mathbb{E}(x_t))^2 = \sum_{j=0}^{t-1} \frac{(\xi_{j+1} + \eta_{j+1}s_j\mathbb{E}(x_j))^2}{(\bar{p}_{j+1}(1 - B_j) + \eta_{j+1}B_j)^2} \cdot (B_j - B_j^2) \cdot \prod_{\ell=j+1}^{t-1} s_\ell^2(1 - B_\ell).$$

Zhu et al. [28] analyzed the Lagrangian problem ($L(\omega)$) via the embedding scheme. They, however, do not succeed to obtain an analytical form of the optimal objective value function $H(\omega)$. Thus, they proposed the prime-dual algorithm to solve the following dual problem of (GMV) numerically,

$$\min_{\omega \in \mathbb{R}_+^{T-1}} H(\omega).$$

In this paper, Proposition 2 does not only derive an analytical policy but also successfully reveal the explicit form of $H(\omega)$. Thus, a simple steepest descent algorithm can be directly applied to derive the optimal Lagrangian multiplier vector ω^* , due to the convexity of $H(\omega)$ (see [28]). Then the optimal strategy of (GMV) can be presented by the portfolio strategy in Proposition 2 with $\omega = \omega^*$. Therefore, our new mean-field formulation clearly, yet powerfully, offers a more efficient and more accurate policy scheme again in this situation, when compared to the existing literature.

Example 1 Consider an example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of U.S market and a bank account. Based on the data provided in Elton et al. [13], the expected values, variances and correlations of the annual return rates of these three indices are given in Table 1.

Table 1: Data for the asset allocation example

	SP	EM	MS
Expected Return	14%	16%	17%
Variance	18.5%	30%	24%
Correlation			
SP	1	0.64	0.79
EM		1	0.75
MS			1

We further assume that the annual risk free rate is 5% ($s_t = 1.05$) and consider a five-period generalized mean-variance model with risk control over bankruptcy, i.e., a (GMV) problem. Then, $\mathbb{E}(\mathbf{P}_t)$, $\text{Cov}(\mathbf{P}_t)$ and $\mathbb{E}(\mathbf{P}_t\mathbf{P}_t')$ can be computed as follows, for $t = 0, 1, \dots, 4$,

$$\mathbb{E}(\mathbf{P}_t) = \begin{bmatrix} 0.09 \\ 0.11 \\ 0.12 \end{bmatrix}, \quad \text{Cov}(\mathbf{P}_t) = \begin{bmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{bmatrix}, \quad \mathbb{E}(\mathbf{P}_t\mathbf{P}_t') = \begin{bmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{bmatrix}.$$

Assume that an investor has initial wealth $x_0 = 1$ and trade-off parameter $\omega_5 = 1$. The disaster level and the acceptable maximum probability of bankruptcy are chosen as $b_t = 0$ and $a_t = 0.10$, respectively, for $t = 1, 2, 3, 4$.

To solve the dual problem of (GMV) and get the optimal Lagrangian multiplier vector ω^* , we consider the following unconstrained problem,

$$\min_{\omega \in \mathbb{R}^4} H(\omega) - \mu \sum_{i=1}^4 \log(\omega_i),$$

where $\sum_{i=1}^4 \log(\omega_i)$ is the barrier function used to ensure $\omega \in \mathbb{R}_+^4$, μ is the barrier parameter and $H(\omega)$ satisfies (15). Theoretically speaking, by setting $\mu \downarrow 0$, we can derive the optimal Lagrangian multiplier vector. Using the steepest descent algorithm, we get

$$\omega^* = [0.0014, 0.2658, 0.2543, 0.0014]'$$

We further have

$$\begin{bmatrix} \bar{p}_1 \\ \bar{p}_2 \\ \bar{p}_3 \\ \bar{p}_4 \\ \bar{p}_5 \end{bmatrix} = \begin{bmatrix} 0.9854 \\ 1.1364 \\ 1.0054 \\ 0.8673 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{bmatrix} = \begin{bmatrix} 0.0615 \\ 0.0550 \\ 0.0256 \\ 0.0001 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \end{bmatrix} = \begin{bmatrix} 0.6200 \\ 0.5828 \\ 0.5513 \\ 0.5250 \\ 0.5 \end{bmatrix}, \quad \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \end{bmatrix} = \begin{bmatrix} 1.0168 \\ 1.0130 \\ 1.0069 \\ 1.0000 \\ 1 \end{bmatrix}.$$

The optimal expected wealth levels are then given by

$$\mathbb{E}(x_0) = 1, \mathbb{E}(x_1) = 1.2366, \mathbb{E}(x_2) = 1.4537, \mathbb{E}(x_3) = 1.6856, \mathbb{E}(x_4) = 1.9353, \mathbb{E}(x_5) = 2.1687.$$

Therefore, according to Proposition 2, the optimal strategy of (GMV) is specified as follows,

$$\begin{aligned} \mathbf{u}_0^* &= (-1.05x_0 + 1.9197)\mathbf{K}, \\ \mathbf{u}_1^* &= (-1.05x_1 + 2.0218)\mathbf{K}, \\ \mathbf{u}_2^* &= (-1.05x_2 + 2.2688)\mathbf{K}, \\ \mathbf{u}_3^* &= (-1.05x_3 + 2.5409)\mathbf{K}, \\ \mathbf{u}_4^* &= (-1.05x_4 + 2.6687)\mathbf{K}, \end{aligned}$$

where

$$\mathbf{K} = \mathbb{E}^{-1}(\mathbf{P}_t \mathbf{P}_t') \mathbb{E}(\mathbf{P}_t) = \begin{bmatrix} 1.0580 \\ -0.1207 \\ 1.1052 \end{bmatrix}.$$

Finally, the variances of the optimal wealth levels are given as

$$\text{Var}(x_1) = 0.1275, \text{Var}(x_2) = 0.1986, \text{Var}(x_3) = 0.2648, \text{Var}(x_4) = 0.3295, \text{Var}(x_5) = 0.3536.$$

We can further get the efficient frontier of (GMV) by adjusting the trade-off parameter ω_5 from 0 to $+\infty$, which is represented by the dash dot line in Figure 1. In the figure, the solid curve above is the efficient frontier of the classical five-period mean-variance model, which is plotted for a comparison purpose.

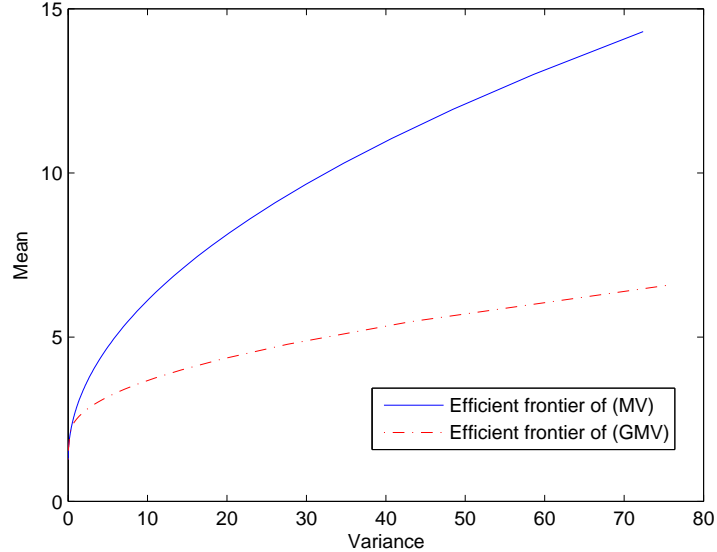


Figure 1: Efficient frontiers of (MV) and (GMV)

5 Conclusions

The nonseparable multi-period mean-variance and related problems have been solved in the literature via embedding scheme, Lagrangian formulation or mean-variance hedging problem. However, we may not be able to derive optimal value functions of these transformed problems analytically, especially, when some constraints are attached to the problem setting. Hence, we often need to invoke some numerical algorithms to compute the corresponding best auxiliary parameter or Lagrangian parameter. In this paper, we adopt the mean-field formulation, as a more efficient means, to directly tackle the nonseparable multi-period mean-variance portfolio selection model, multi-period mean-variance model with intertemporal restrictions, and generalized mean-variance model with risk control over bankruptcy. Under this newly proposed framework of mean-field formulations, we are capable of deriving analytical solutions for all these problems, thus improving the solution quality and facilitating the solution process.

References

- [1] D. Andersson, B. Djehiche, A maximum principle for SDEs of mean-field type, *Applied Mathematics and Optimization*, 63 (2011), 341-356.
- [2] V.S. Borkar, K.S. Kumar, McKean-Vlasov limit in portfolio optimization, *Stochastic Processes and Their Applications*, 28 (2010), 884-906.
- [3] R. Buckdahn, B. Djehiche, J. Li, A general stochastic maximum principle for SDEs of mean-field type, *Applied Mathematics and Optimization*, 64 (2011), 197-216.
- [4] R. Buckdahn, J. Li, S. Peng, Mean-field backward stochastic differential equations and related partial differential equations, *Stochastic Processes and their Applications*, 119 (2009), 3133-3154.

- [5] U. Çelikyurt, S. Özekici, Multi-period portfolio optimization models in stochastic markets using the mean-variance approach, *European Journal of Operational Research*, 179 (2007), 186-202.
- [6] A. Černý, J. Kellsen, Hedging by sequential regressions revisited, *Mathematical Finance*, 19 (2009), 591-617.
- [7] T. Chan, Dynamics of the McKean-Vlasov equation, *Annals of Probability*, 22 (1994), 431-441.
- [8] P. Chen, H.L. Yang, Markowitz's mean-variance asset-liability management with regime switching: A multi-period model, *Applied Mathematical Finance*, 18 (2011), 29-50.
- [9] M.C. Chiu, D. Li, Asset and liability management under a continuous-time mean-variance optimization framework, *Insurance: Mathematics and Economics*, 39 (2006), 330-355.
- [10] O.L.V. Costa, R.B. Nabholz, Multi-period mean-variance optimization with intertemporal restrictions, *Journal of Optimization Theory and Applications*, 134 (2007), 257-274.
- [11] D. Crisan, J. Xiong, Approximate McKean-Vlasov representations for a class of SPDEs, *Stochastics*, 82 (2010), 53-68.
- [12] D.A. Dawson, Critical dynamics and fluctuations for a mean-field model of cooperative behavior, *Journal of Statistical Physics*, 31 (1983), 29-85.
- [13] E.J. Elton, M.J. Gruber, S.J. Brown, and W.N. Goetzmann, *Modern Portfolio Theory and Investment Analysis*, John Wiley & Sons, (2007).
- [14] M. Kac, Foundations of kinetic theory, *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, 3 (1956), 171-197.
- [15] M. Leippold, F. Trojani, and P. Vanini, A geometric approach to multi-period mean-variance optimization of assets and liabilities, *Journal of Economic Dynamics and Control*, 28 (2004), 1079-1113.
- [16] D. Li, W.L. Ng, Optimal dynamic portfolio selection: Multi-period mean-variance formulation, *Mathematical Finance*, 10 (2000), 387-406.
- [17] X. Li, X.Y. Zhou, A.E.B. Lim, Dynamic mean-variance portfolio selection with no-shorting constraints, *SIAM Journal on Control and Optimization*, 40 (2002), 1540-1555.
- [18] H.M. Markowitz, Portfolio selection, *Journal of Finance*, 7 (1952), 77-91.
- [19] H.P. McKean, A class of Markov processes associated with nonlinear parabolic equations, *Proceedings of the National Academy of Sciences of the United States of America*, 56 (1966), 1907-1911.
- [20] R.C. Merton, An analytic derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis*, 7 (1972), 1851-1872.
- [21] T. Meyer-Brandis, B. Oksendal, X. Y. Zhou, A mean-field stochastic maximum principle via Malliavin calculus, *A special issue for Mark Davis' Festschrift, to appear in Stochastics*, (2011).

- [22] M. Nourian, P.E. Caines, R.P. Malhamé, M. Huang, Nash, social and centralized solutions to consensus problems via mean field control theory, *to appear IEEE Transaction on Automatic Control*, (2012).
- [23] M. Schweizer, Approximation pricing and the variance-optimal martingale measure, *Annals of Probability*, 24 (1996), 206-236.
- [24] W.G. Sun, C.F. Wang, The mean-variance investment problem in a constrained financial market, *Journal of Mathematical Economics*, 42 (2006), 885-895.
- [25] J.M. Xia, J.A. Yan, Markowitz's portfolio optimization in an incomplete market, *Mathematical Finance*, 16 (2006), 203-216.
- [26] J.M. Yong, A linear-quadratic optimal control problem for mean-field stochastic differential equations, *Working paper*, arXiv:1110.1564, (2012).
- [27] X.Y. Zhou, D. Li, Continuous-time mean-variance portfolio selection: A stochastic LQ framework, *Applied Mathematics and Optimization*, 42 (2000), 19-33.
- [28] S.S. Zhu, D. Li, S.Y. Wang, Risk control over bankruptcy in dynamic portfolio selection: A generalized mean-variance formulation, *IEEE Transactions on Automatic Control*, 49 (2004), 447-457.